ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, No. 1, 1976

GENERALIZED QUADRANGLES HAVING A PRIME PARAMETER†

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ABSTRACT

Generalized quadrangles $\underset{\sim}{\mathcal{Q}}$ are studied in which s or t is prime and Aut $\underset{\sim}{\mathcal{Q}}$ has rank 3 on points.

1. Introduction

A generalized quadrangle $\underset{\sim}{\mathcal{Q}}$ of order (s, t) consists of a set of points and lines, with each line on $s+1$ points and each point on $t+1$ lines, such that two points are on at most one line and a point not on a line is collinear with exactly one point of the line. We will study the case where s or t is prime and Aut $\underset{\sim}{\mathcal{Q}}$ has rank 3 on points.

THEOREM 1.1. *Let $\underset{\sim}{\mathcal{Q}}$ be a generalized quadrangle of order (p, t) with p prime and $t > 1$. Suppose $G =$ Aut $\underset{\sim}{\mathcal{Q}}$ has rank 3 on points. Then either $t = p^2 - p - 1$ and $p^3 \nmid |G|$, or $G \cong PSp(4, p)$ or $P\Gamma U(4, p)$ and $\underset{\sim}{\mathcal{Q}}$ is one of the usual quadrangles associated with these groups, or $p = 2$, $G = A_6$ and $\underset{\sim}{\mathcal{Q}}$ is one of the usual quadrangles associated with $PS_p(4, 2)$.*

A group G having a BN-pair whose Weyl group is D_8 naturally acts as an automorphism group of a generalized quadrangle of order (s, t) with $s > 1$ and $t > 1$. Moreover, $(1+s)(1+t)(1+st)s^2t^2$ divides $|G|$. Thus, as an immediate consequence of (1.1) we have:

COROLLARY 1.2. *Let G be a finite group having BN-pair and Weyl group D_8. Suppose that $|P : B| - 1$ is a prime p for some maximal parabolic subgroup P. Then G has a normal subgroup H isomorphic to $PSp(4, p)$ or $PSU(4, p)$, with the usual BN-pair induced on H.*

† This research was supported in part by NSF grant GP 37982X.

Received November 14, 1974

COROLLARY 1.3. *Let G be a rank 3 group having subdegrees* 1, $p\gamma$, p^2 *with p a prime,* $p \nmid \gamma\delta$, $(\gamma, \delta) = 1$, *r a power of p,* $r > 1$ *and either* $(1+\delta)r \geqq \gamma$ *or* $p = 2$ *and* $\delta = 1$. *Then G can be regarded as acting on the singular points of a symplectic or orthogonal geometry over* $GF(p)$, *or on the singular lines of a* 4*-dimensional symplectic or unitary geometry over* $GF(p)$.

Corollary 1.3 is a consequence of (1.1) and Kantor [4]. Further consequences of the preceding sort also follow from the latter paper. The present work originated in an attempt to push the rather elementary methods of [4] somewhat further. The proof of (1.1) requires little more than elementary group theory, combined with results of Higman [1], [2], [3]. The case $t = p$ is especially simple; for both this reason, and later convenience, it has been presented separately in Section 4.

The basic idea is to take a Sylow p-subgroup P of G, and then see how both its center and various point-and line-stabilizers in P must behave. The same methods yield the following result; the details are left to the reader.

THEOREM 1.4. *Let* $\mathcal{Q}$ *be a generalized quadrangle of order* (s, p) *with p prime and* $s > 1$. *Suppose* $G = \operatorname{Aut} \mathcal{Q}$ *has rank* 3 *on points,* $p^3 \mid |G|$, *and either* $s \neq p^2 - p - 1$ *or* $p^4 \mid |G|$. *Then* $G \cong PSp(4, p)$ *or* $P\Gamma U(4, p)$, *and* $\mathcal{Q}$ *is one of the usual quadrangles associated with these groups.*

We remark that there is a well-known quadrangle of order (3, 5) for which $3^3 \mid |\operatorname{Aut} \mathcal{Q}|$ (see, e.g., Higman [2], p. 287); $\operatorname{Aut} \mathcal{Q}$ has rank 3 on points and rank 5 on lines.

Finally, we note that the methods presented here apply to other situations, such as rank 4 automorphism groups of generalized hexagons of order (p, p) with p prime.

2. Preliminary results

Let $\mathcal{Q}$ be a generalized quadrangle of order (s, t). If x is a point, $\Gamma(x)$ denotes the set of points y such that a line xy exists, $x^\perp = \{x\} \cup \Gamma(x)$, and $\Delta(x)$ is the complement of $x^\perp$. We call x and y joined or adjacent if xy exists; and dually lines L and M are adjacent if $L \cap M$ is a point.

$H(x)$ will denote the set of elements of $H \leqq \operatorname{Aut} \mathcal{Q}$ fixing each line on x, while $H(L)$ is the pointwise stabilizer of L.

LEMMA 2.1. *Let* $\mathcal{Q}$ *be a generalized quadrangle of order* (s, t).

(i) *Suppose a subgroup H of* $\operatorname{Aut} \mathcal{Q}$ *fixes at least three points of some line and* *at least three lines through some point. If no fixed point H is joined to all others, and no fixed line meets all others, then the set of fixed points and lines of H form a sub-quadrangle of order (s', t') for some $s' \leqq s$ and $t' \leqq t$.*

(ii) *If $\underset{\sim}{\mathcal{Q}}$ has a proper subquadrangle of order (s, t'), then $t \geqq st'$.*

(iii) *$t^2 \geqq s$ and $s^2 \geqq t$ if $s > 1$ and $t > 1$.*

PROOF. (i) is straightforward. To prove (ii) (which is due to Payne [6] and Thas [7]), take x outside of the subquadrangle $\underset{\sim}{\mathcal{Q}}_1$. Then each of the $t+1$ lines through x meets $\underset{\sim}{\mathcal{Q}}_1$ at most once. Counting in two ways the pairs (y, L) with $y \in L$, x and y collinear, and y, $L \in \underset{\sim}{\mathcal{Q}}_1$, we find that $(t+1)(t'+1) \geqq 1 + (s+1)t' + st'^2$ (the latter being the number of lines of $\underset{\sim}{\mathcal{Q}}_1$). This implies that $t \geqq st'$.

Finally, (iii) is Higman's inequality [2].

The second part of the following transitivity-boosting lemma is probably well-known; the proof of the first part has the same flavor as the one in Kantor [4].

LEMMA 2.2. *Suppose $G \leqq \mathrm{Aut}\,\underset{\sim}{\mathcal{Q}}$ has rank 3 on points. Then*

(i) *G_x is 2-transitive on the lines through x; and*

(ii) *If $(s, t+1) = 1$ and $y \in \Gamma(x)$, then G_{xy} is transitive on $y^\perp - xy$.*

PROOF. (i) Let $x \in L$. Then G_{xL} contains a Sylow p-subgroup P of G_x for each prime $p \mid t$. It suffices to show that for each p and P, each orbit L'^P of lines $\neq L$ on x has length divisible by t_p (the p-part of t).

Suppose $|L'^P| < t_p$ for some such orbit. There exist points $y \in L - \{x\}$ and $y' \in L' - \{x\}$ whose $P_{L'} = P_{LL'}$ orbits have lengths $\leqq s_p$. Thus, $|P_{L'yy'}| \geqq |P_{L'}|/s_p^2 > |P|/s_p^2 t_p$, so $|P^*: P_{yy'}| < s_p^2 t_p = |\Delta(y)|_p$ for a Sylow p-subgroup $P^* \geqq P_{yy'}$ of G_y. Since $y' \in \Delta(y)$ and G_y is transitive on $\Delta(y)$, this is impossible.

(ii) Since $(|\Gamma(x)|, |\Delta(x)|) = (s(t+1), s^2 t) = s$, each G_{xy}-orbit on $\Delta(x)$ has length divisible by $s^2 t/s = |y^\perp - xy|$.

REMARK. Note that the hypotheses of (2.2) guarantee that G_L is 2-transitive on L. What (2.2) says is that a second 2-transitive group is also always available.

LEMMA 2.3. *The pointwise stabilizer $G(x^\perp)$ of $x^\perp$ is semiregular on $\Delta(x)$, and $|G(x^\perp)| \mid t$.*

PROOF. The first statement is (6.17) of Higman [2], and follows immediately from (2.1 i). To prove the second one, let M be a line not on x, and set $\{y\} = x^\perp \cap M$. Then each $u \in x^\perp - xy$ is joined to some $w \in M - \{y\}$, and hence $G(x^\perp)_M \leqq G(x^\perp)_w = 1$.

THEOREM 2.4. (Higman [1].) *Assume* $G \leqq \mathrm{Aut}\, \underset{\sim}{\mathcal{Q}}$ *has rank* 3 *on points, and* $s = t = |G(x^{\perp})|$. *Then* $\underset{\sim}{\mathcal{Q}}$ *is isomorphic to the usual quadrangle for* $Sp(4, s)$, *and* $G \geq PSp(4, s)$.

THEOREM 2.5. (Higman [3].) *Assume* $G \leqq \mathrm{Aut}\, \underset{\sim}{\mathcal{Q}}$ *has rank* 3 *on points,* $s = t^2$ *and* $|G(x^{\perp})| = t$. *Then* $\underset{\sim}{\mathcal{Q}}$ *is isomorphic to the usual quadrangle for* $PSU(4, t)$, *and* $G \geq PSU(4, t)$.

LEMMA 2.6. (Higman [2, (6.1)].) $s^2(1 + st)/(s + t)$ *is an integer.*

COROLLARY 2.7. *Suppose* $(s, t) = 1$, $s > 1$ *and* $t > 1$.
(i) *If* $s \mid t \pm 1$ *then* $t = s^2 - s - 1$.
(ii) *If* $s \mid t - 3$ *and* $3 \mid s - 1$ *then* $t = 2s + 3$.
(iii) *If* $s \mid t - 2$ *then* $t = s + 2$.

PROOF. We will prove (ii); (i) and (iii) are similar. By (2.6), $s + t \mid s^2 - 1$. We can write $s^2 - 1 = \alpha(s + t)$ and $t - 3 = \beta s$ for integers α and β. Then $-1 \equiv 3\alpha \pmod s$, so $\alpha \equiv (s - 1)/3 \pmod s$. Write $\alpha = ((s - 1)/3) + s\gamma$. Then $s^2 - 1 = (((s - 1)/3) + s\gamma)(s + t)$ implies that $\gamma = 0$ and $3(s + 1) = s + t$, as required.

3. Hyperbolic lines

Let $\mathcal{G}$ be any strongly regular graph with parameters n, k, l, λ, μ. For each point x, $\Gamma(x)$ will denote the set of points joined to x, and $\Delta(x)$ the set of points $\neq x$ not joined to x. Write $x^{\perp} = \{x\} \cup \Gamma(x)$. The line xy, $x \neq y$, is defined by

$$xy = \bigcap \{w^{\perp} \mid x, y \in w^{\perp}\} = \bigcap \{w^{\perp} \mid w \in x^{\perp} \cap y^{\perp}\}. \tag{3.1}$$

This line is called *singular* if $y \in \Gamma(x)$ and *hyperbolic* if $y \in \Delta(x)$.

LEMMA 3.2. (Higman [2, p. 282].)
(i) *Two adjacent points are on a unique singular line.*
(ii) *Two non-adjacent points are on at most one hyperbolic line, and are on no singular line, if* $\mathcal{Q}$ *is the point-graph of a generalized quadrangle.*

Consider the following hypothesis:

(H) *Each hyperbolic line has* $h + 1$ *points, and two distinct lines meet at most once.*

This will be the case, for example, if (3.2ii) holds and Aut $\mathcal{G}$ is transitive on pairs of non-adjacent points.

LEMMA 3.3. *Assume* (H). *Then the following hold.*

(i) *x is on l/h hyperbolic lines.*

(ii) *There are $nl/h(h+1)$ hyperbolic lines.*

(iii) $h \mid k-\lambda-1$.

(iv) *If $w \in \Delta(x)$ then w is on $l/h-(k-\mu+1)$ hyperbolic lines missing $x^\perp$.*

(v) *There are $l[l/h-(k-\mu+1)]/(h+1)$ hyperbolic lines missing $x^\perp$.*

PROOF. (i) and (ii) are easy. If $y \in \Gamma(x)$ then $y^\perp \cap \Delta(x)$ is a union of hyperbolic lines with x removed; this implies (iii).

To prove (iv), note that w is joined to μ points of $\Gamma(x)$. Let y be any of the remaining $k-\mu$ points of $\Gamma(x)$. If wy meets $\Gamma(x)$ at a second point $y' \neq y$, then by (H), $y' \in \Delta(y)$ and $wy = yy'$. But now, $y, y' \in x^\perp$ implies that $yy' \subseteq x^\perp$, and hence that $w \in x^\perp$.

Thus, w is on exactly $k-\mu$ hyperbolic lines meeting $x^\perp$. By (i), this proves (iv).

Finally, count the pairs (w, L) with $w \in \Delta(x) \cap L$, L a hyperbolic line, and $L \cap x^\perp = \phi$, in order to obtain (v).

COROLLARY 3.4. *If* (H) *holds, and* Aut $\mathscr{G}$ *is transitive on hyperbolic lines, then each hyperbolic line misses exactly $l-h(k-\mu+1)$ sets $x^\perp$.*

PROOF. By (3.3), the desired number is

$$n \cdot l[l/h-(k-\mu+1)](h+1)^{-1} \cdot (nl/h(h+1))^{-1}.$$

LEMMA 3.5. *If* (H) *and* (3.2ii) *hold, then*

(i) *$x^\perp$ contains $s^2t(t+1)/h(h+1)$ hyperbolic lines; and*

(ii) *$|G(x^\perp)|$ divides h.*

PROOF.

(i) Count the pairs (y, H) with $y \in H \subset x^\perp$ and H a hyperbolic line.

(ii) Higman [2, (6.17)].

4. The case $s = t = p$

Theorem 1.1 is particularly easy when $s = t = p$ is prime. We may assume $p > 2$. Let P be a Sylow p-subgroup of G. Then P fixes some x and some (singular) line L on x. Moreover, P is transitive on $L - \{x\}$, $\Delta(x)$ and $x^\perp - L$ (by (2.2)). Set $Z = Z(P) \cap P(x) \cap P(L)$. Since $p^3 = |\Delta(x)| \mid |G|$, $Z \neq 1$.

Let $w \in \Delta(x)$, and suppose $P_w \neq 1$. Then $P_w = P(wy)$ if $y \in L \cap \Gamma(w)$. If now Z is transitive on the lines $\neq L$ on y, then $P_w \leqq G(y^\perp)$ and Higman's result (2.4) applies. Assume next that $Z \leqq G(y)$. Then the transitivity of P shows that Z fixes every line meeting L. Hence, Higman's result (2.4) applies to the dual of $\mathcal{Q}$ if G has rank 3 on lines. But by (2.2), if G does not have rank 3 on lines, then $|K^P| \leqq p^2$ for a line K on w. This implies that $|P_K| \geqq p^2$, so $P_{Kw} \neq 1$. Then, by (2.1), the set of fixed points and lines of P_{Kw} form a subquadrangle of order (p, p), which is absurd.

Thus, we may assume $|P| = p^3$. Then no nontrivial p-element can fix two nonadjacent points. In particular, $P(L) = P_y$ is regular on $x^\perp - L$. (Also, P is regular on $\Delta(x)$, so G has rank 3 on lines.) Since $|P(x)| = p^2$, we see that $P(x)$ has $p+1$ subgroups of order p, each fixing a unique line on x pointwise. Hence, by the Frattini argument, $N(P(x))_x$ is 2-transitive on these $p+1$ subgroups, and hence induces at least $\mathrm{SL}(2, p)$ on $P(x)$.

Moreover, $|Z| = p$ here, and $Z = P(x) \cap P(L)$. Thus, $Z \leqq P(y)$ would again permit (2.4) to be applied to the dual of $\mathcal{Q}$. It follows as above that $N(P(L))_L$ is 2-transitive on the $p+1$ subgroups of order p of $P(L)$, and induces at least $SL(2, p)$ on $P(L)$.

In view of the action of $N(P(x))_x$ on $P(x)$, there is a 2-element $t \in N(P(x))_x \cap N(P(L))$ which inverts $P(x)$ and centralizes $P(L)/Z$. Then t normalizes each of the $p+1$ subgroups of $P(x)$ corresponding to the lines on x, and hence $t \in G(x)$. Similarly, there is a 2-element $t' \in N(P(L))_L \cap N(P(x))$ which inverts $P(L)$ and centralizes $P(x)/Z$. By Sylow's theorem, we may assume that $\langle t, t' \rangle \leqq N(P(x)) \cap N(P(L))$ is a 2-group.

Now tt' centralizes Z and inverts P/Z and tt' fixes some line $L_1 \neq L$ on x. Then also tt' fixes one of the p points of $L_1 - \{x\}$, and the transitivity of Z on $L_1 - \{x\}$ shows that $tt' \in G(L_1)$. Dually, $tt' \in G(y)$ for some $y \in L - \{x\}$. (Recall that Z is transitive on the lines $\neq L$ on y.) Thus, (2.1i) implies that the set of fixed points and lines of tt' is a subquadrangle of order (p, p). This is ridiculous, and the case $s = t = p$ is completed.

5. The case $s = p$ and $p^3 \mid |G|$

Let $\mathcal{Q}$ and G be as in Theorem 1.1. Let P be a Sylow p-subgroup of G. Then P fixes some point x. Set $Z = Z(P)$.

It is easy to handle the case $p = 2$ (since $t \leqq p^2$ by (2.1)). We may thus assume $p > 2$. By Section 4, we may also assume $p \neq t$.

Throughout this section we will assume $p^3 \mid |G|$.

LEMMA 5.1. $t > p$.

PROOF. Suppose $t<p$. Then $P \leqq G(x)$. As $|\Delta(x)|=p^2t$, $P_w \neq 1$ for some $w \in \Delta(x)$. Certainly, $P_w = P(wy)$ for each $y \in x^\perp \cap w^\perp$. By (2.1i), the set of fixed points and lines of P_w form a subquadrangle of order (p,t), which is absurd.

LEMMA 5.2. $p \mid t$.

PROOF. Suppose $p \nmid t$. By (2.1) and (5.1), $p<t<p^2$. Also, for some $w \in \Delta(x)$, $P_w \neq 1$ and P_w is Sylow in G_{xw}.

Consider first the possibility $p \mid t+1$. Here no nontrivial subgroup of P can fix elementwise a subquadrangle of $\mathcal{Q}$. For, by (2.1) such a quadrangle would have order (p,t_1) with $pt_1 \leqq t<p^2$ and $p \mid t_1+1$, so $t_1=p-1$. However, by (2.6) no quadrangle of order $(p,p-1)$ can exist.

On the other hand, $|P_K| \geqq p^2$ for one of the pt^2 lines K not on x. Then $P(K) \neq 1$, and we may assume $w \in K$. Now $P(K)$ fixes at least p lines L' on x, and at least p on w. Since w is joined to some point of $L'-\{x\}$, this contradicts (2.1) and the preceding paragraph.

From now on we may assume $p \nmid t+1$. Then p fixes some line L on x. Moreover, the set $\mathcal{Q}_1$ of fixed points and lines of P_w from a subquadrangle, necessarily of order (p,t_1) for some $t_1 \geqq 1$. Here $t_1 \equiv t \pmod p$, while $pt_1 \leqq t < p^2$ by (2.1). Also, since P_w is Sylow in G_{xw}, $N(P_w)$ is transitive on the ordered pairs of non-adjacent points of $\mathcal{Q}_1$.

We claim that $|P|=p^3$. For suppose $|P| \geqq p^4$. Then $1 \neq P_{wL'} < P_w$ for some line L' on x. The set of fixed points and lines of $P_{wL'}$ forms a subquadrangle $\mathcal{Q}_2 \supset \mathcal{Q}_1$ of $\mathcal{Q}$ of order (p,t_2) for some t_2. By (2.1), $p^2t_1 < pt_2 < t < p^2$, which is impossible.

Thus, $|P|=p^3$ and $|P_w|=p$. But the transitivity of $N(P_w)$ implies that $p^3 \mid |N(P_w)|$. Hence $P_w \leqq Z(P)$.

Since $|x^\perp - L| = pt \not\equiv 0 \pmod{p^2}$, $|P_u| \geqq p^2$ for some $u \in x^\perp - L$. Then P_u is not conjugate in G to any P_w, so P_u fixes no point of $x^\perp - xu$. Thus, $Z(P)$ fixes xu. There are thus exactly t_1+1 lines xu with $|P(xu)| \geqq p^2$. If v is any point of $x^\perp$ not on any of these lines, then $|v^P| < pt < p$, so $P_v \neq 1$ and $Z(P) \leqq C(P(xv))$ implies that $P(xv)$ fixes a second line on x pointwise, and hence determines a subquadrangle of order (p,t_2), say. But this time, $p \leqq t_2$, and this contradicts (2.1).

By (5.2), we now know P fixes some line L on x. Let t_p denote the p-part of t.

LEMMA 5.3. *If $p^2t_p^2$ divides $|G|$, then the conclusions of* (1.1) *hold.*

PROOF. By (5.2), $p \mid t$. Then $|P| \geqq p^4$, and $|P| \geqq p^6$ if $t=p^2$. By (2.1), $t \leqq p^2$.

We have $|\Delta(x)| = p^2t \equiv 0 \pmod{p^3}$. Let $w \in \Delta(x)$. Then $p^4 \geqq p^2t \geqq |w^P| \geqq p^3$, so $|w^P|$ is p^2t_p. In particular, $|P_w| \geqq t_p$. Note that $P_w = P(yw)$ if $\{y\} = L \cap w^\perp$.

We claim that P_w fixes no point of $\Delta(y)$. For otherwise, by (2.1) P_w fixes elementwise a subquadrangle of order (p, t_1), where $pt_1 \leqq t \leqq p^2$ and $p \mid t_1$. Thus, $t = p^2$, so $|P_w| \geqq p^2$. Now $t - t_1 < p^2$ implies that, for some line $M \neq L$ on x, $P_w > P_{wM} \neq 1$. Then P_{wM} fixes more than $p+1$ lines through x; by (2.1), it determines a subquadrangle of order (p, t_2) with $pt_2 \leqq t = p^2$ and $t_2 > t_1$. This contradiction proves our claim.

Thus, P_w fixes only points of $y^\perp$. Since w and y are arbitrary, $Z = Z(P)$ fixes each point of L.

Let $u \in x^\perp - L$. Since $pt \leqq p^3$, by (2.2) each P-orbit on $x^\perp - L$ has length pt_p. Thus, $|P: P_u| = pt_p$. Clearly, P_u has an orbit $\neq \{xu\}$ of lines K on u of length $\leqq t_p$. Thus, $|P: P_K| = |P: P_{uK}| \leqq Pt_p^2$, so $P_K \neq 1$.

We claim that all fixed lines of P_K are adjacent to xu. For otherwise, by (2.1) the set $\underset{\sim}{\mathcal{Q}}_1$ of fixed points and lines of P_K is a subquadrangle of order (p, t_1) (as $P_K \leqq P(xu)$ fixes at least $p+1$ lines on x). Here $p^2 \geqq t \geqq pt_1$ by (2.1), while $p \mid t_1$. Thus, $t = p^2$ and $t_1 = p$. By (2.1), P_K must be semiregular on the $t - t_1$ lines through x it moves, so $|P_K| = p$. Thus, $|K^P| \geqq p^5$, so K^P consists of all lines not adjacent to L. Moreover, $N_P(P_K)$ is transitive on $K^P \cap \underset{\sim}{\mathcal{Q}}_1$, and hence (by intersecting these lines with $x^\perp$) also on $(x^\perp - L) \cap \underset{\sim}{\mathcal{Q}}_1$. Since L can be any line of $\underset{\sim}{\mathcal{Q}}_1$, it follows that $N(P_K)$ has rank 3 on the dual of $\underset{\sim}{\mathcal{Q}}_1$. Moreover, $p^4 \nmid |N(P_K)^{\mathcal{Q}_1}|$ since $P_K^{\mathcal{Q}_1} = 1$. By Section 4, this is impossible, and our claim is proved.

Thus, $Z \leqq C(P_K)$ must fix xu. As $u \in x^\perp - L$ was arbitrary, we now have $Z \leqq P(x) \cap P(L)$.

Let $G(L^\perp)$ denote the set of elements of G fixing every line adjacent to L. Suppose that $Z \cap G(L^\perp) \neq 1$. By (2.3) (applied to the dual of $\underset{\sim}{\mathcal{Q}}$), $|G(L^\perp)| \mid p$. Thus, $G(L^\perp) \leqq Z$. Clearly, $G(L^\perp) \trianglelefteq G_L$. Set $E = \langle G(M^\perp) \mid x \in M \rangle$. Then $E \trianglelefteq G(x)$ is elementary abelian, and G_x acts 2-transitively on the $t+1 > p+1$ groups $G(M^\perp)$. In particular, $|E| \geqq p^3$. But $GL(3, p)$ has no such 2-transitive subgroup since $t + 1 < p^2 + p + 1$ (Mitchell [5]). Thus $|E| \geqq p^4$. If now $t < p^2$ then $|P| \geqq p^5$. Then $|P_w| \geqq p^2$, so $P_w > P_{wK} \neq 1$ for some line K adjacent to yw. (Note that $|P_w| \not\leqq |G((yw)^\perp)|$.) As usual, P_{wK} determines a subquadrangle, and (2.1) produces a contradiction. Thus, $t = p^2$, so $|xu^P| = p^2$. By (2.5), we may assume that G does not have rank 3 on lines. Then $|K^P| \leqq p^4$ for each line K not adjacent to L, so $|P_K| \geqq p^2$. As usual, (2.1) implies that for $w \in K \cap \Delta(x)$, the set of fixed points and lines of P_{Kw} form a quadrangle of order (p, p). Hence, again by (2.1), $|P_{Kw}| = p$, $|P_K| \leqq p^2$, and hence $|P| = p^6$. Now $|P: P(x)| \leqq p^2 = |xu^P| = t$ shows that no subgroup of P can fix exactly $p+1$ lines on x, whereas P_{Kw} is such a subgroup.

Thus, we may assume that $Z \cap G(L^\perp) = 1$, and (eventually) will derive a contradiction from this assumption. Since P is transitive on $L-\{x\}$, $Z\cap P(y)=1$ for each $y\in L-\{x\}$. Since $P(L)$ is Sylow in $G(L)$, we can find $g\in G_L$ such that $P^g\geqq P(L)$ and P^g is Sylow in G_{yL}. Set $W=Z^g$. Then $W\leqq P(L)$. Moreover, $P_w\leqq P(L)\leqq C_P(W)$.

Recall that all fixed points of P_w are in $y^\perp$. Since P_w fixes L and wy pointwise, while $N(P_w)$ is transitive on ordered pairs of non-adjacent fixed points of P_w, we must have $|N:P_w|\geqq|L-\{y\}|\cdot|wy-\{y\}|=p^2$, where $N=N_P(P_w)$.

We can now prove $t=p^2$. For suppose $t<p^2$. By (2.1), P_w is semiregular on the lines $\neq L$ through x, so $|P_w|=p$ and $|P|=p^4$. In particular, $N=C_P(P_w)$ and $|P:N|\leqq p$. Also, $P_w\not\leqq P(x)$ implies that $P_w\not\leqq Z$, so $|N|=p^3$. Then $P_wZ\leqq Z(N)$ implies that N is abelian. Hence, N centralizes its subgroup W. But the transitivity of $N(P_w)$ implies that N is transitive on $L-\{x\}$. Thus, $W\leqq P(y)$ fixes every line meeting $L-\{x\}$. Since Z is conjugate to W, Z must fix every line meeting $L-\{y\}$, which is not the case.

Thus, $t=p^2$ and $|P|\geqq p^6$.

Next note that $P(x^\perp)=1$. For otherwise, h is a power of p by (3.3), so $h=p^2$ by (3.5i), whereas $s^2t/h\geqq(s-1)(t+1)+1$ by (3.3iv).

Hence, the transitivity of P on $x^\perp-L$ (see (2.2)) implies that Z is semiregular on $x^\perp-L$. Thus, for each L' on x, $P(x)\cap P(L')$ contains a G_x-conjugate $Z'\neq Z$ of Z. In fact, if P' is a Sylow p-subgroup of $G_{xL'}$ such that $P'(x)=P(x)$, then we can choose $Z'=Z(P')$. Thus, $Z(P(x))$ has p^2+1 nontrivial subgroups, any two meeting trivially. In particular, $|Z(P(x))|\geqq p^3$. But $\langle P,P'\rangle$ permutes p^2+1 such subgroups 2-transitively, so $|Z(P(x))|\geqq p^4$.

If $|P(x)|\geqq p^5$, then $P(x)_w\neq 1$, and this contradicts (2.1).

Thus, $|P(x)|=p^4$ and $P(x)$ is elementary abelian. Moreover, $|P(x)\cap P(L)|=p^3$. Since $P(x)$ is transitive on $L-\{x\}$ and centralizes $P(x)\cap P(y)$, we have $P(x)\cap P(y)\leqq P(L^\perp)=1$. Thus, since $|P(y)\cap P(L)|=p^3$, necessarily $|P(L)|\geqq p^3\cdot p^3$, so $|P|\geqq p^7$ and $|P_w|\geqq p^3$. Consequently, $P_{wM}\neq 1$ for some $M\neq L$ on x. By (2.1), $P_{wM}\cap P(x)=1$.

$N(P(x))$ induces the same 2-transitive representation on the p^2+1 lines on x and the p^2+1 subgroups $P(x)\cap P(L)$ of $P(x)$. It thus induces a subgroup of $GL(4,p)$, 2-transitive on p^2+1 hyperplanes, and having a nontrivial p-subgroup (induced by P_{wM}) fixing more than one such hyperplane. However, $GL(4,p)$ has no such subgroup.

Proof of Theorem 1.1 when $p^3\mid|G|$

In view of the preceding lemmas, it remains to eliminate the case $p\mid t$, $p<t$,

and $p^2t_p^2 \nmid |G|$. By (2.1iii), either $t < p^2$ and $|P| = p^3$, or $t = p^2$ and $|P| = p^4$ or p^5.

Suppose first that $t < p^2$. Then P is semiregular on $\Delta(x)$. Hence, if $y \in L - \{x\}$ then $P_y = P(L)$ is semiregular on $x^\perp - L$. Consequently, if $u \in x^\perp - L$, then P_u (which is nontrivial as otherwise $p^3 = |u^P| \leqq |x^\perp - L| = pt$) is semiregular on $x^\perp - xu$. In particular, $Z = Z(P) \leqq G(x)$. By (2.2), $Z < P$, so $|Z| = p$. But $P(L) \lhd P$, so $Z \leqq P(L)$. Thus, $Z = P(L)_{L'}$ whenever $x \in L' \neq L$. Consequently, P_u and Z are conjugate in G_x (by (2.2)), so $P_u = P(x) \cap P(xu)$. Now $P(x)$ has $t + 1 > p + 1$ distinct proper subgroups, so $|P(x)| \geqq p^3 = |P|$. By (2.2ii), this is impossible.

Thus, $t = p^2$. Suppose next that $|P| = p^4$. Then once again, P is semiregular on $\Delta(x)$, $P_u \neq 1$ for each $u \in x^\perp - L$, P_u is semiregular on $x^\perp - xu$, and $Z \leqq G(x)$. Moreover, $|Z \cap P(L)| = p = |P_u|$ by the semiregularity of $P(L)$, and $P_u = P(xu)_L$. Thus, $Z \cap P(L) = P(L)_{L'}$ whenever $x \in L' \neq L$. As above, we then have $P_u = P(xu)_L$ conjugate to $Z \cap P(L)$, so $P_u \leqq P(x)$, $|P(x)| \geqq p^3$, and hence $|P: P(x)| \leqq p$. Once again, this contradicts (2.2ii).

Consequently, $|P| = p^5$. Now $|P_w| = p$ for each $w \in \Delta(x)$, while $|P_u| = p^2$ for each $u \in x^\perp - L$. Thus, P_u fixes no points of $x^\perp - xu$, so $Z \leqq P(x)$ once again. Also, $Z \cap P(L) \neq 1$. Since $P(x^\perp) = 1$ as in the proof of (5.3), $Z \cap P(L)$ is semiregular on $x^\perp - L$. Thus, $|Z \cap P(L)| = p$.

For each $u \in x^\perp - L$, $Z(P(x)) \cap P(xu)$ contains a G_x-conjugate of $Z \cap P(L)$. Thus, $Z(P(x))$ has $p^2 + 1$ such subgroups, and $|Z(P(x))| \geqq p^3$. Since $N(P(x))$ permutes these subgroups 2-transitively, $|Z(P(x))| \geqq p^4$. But now $|P: P(x)| \leqq p$ is again ridiculous.

This completes the proof of (1.1) when $p^3 \mid |G|$.

6. The case $p^3 \nmid |G|$

We now consider the case $p^3 \nmid |G|$ of Theorem 1.1. Certainly, $p^2 \mid |G|$ since $|\Delta(x)| = p^2 t$. Thus, a Sylow p-subgroup P of G has order p^2, and fixes some point x. By (2.7), $p \nmid t + 1$, so P fixes $1 + \varepsilon \geqq 2$ lines on x. Let L be such a line. P is semiregular on $\Delta(x)$, so $P(L)$ is semiregular on $x^\perp - L$.

LEMMA 6.1. *$\varepsilon = 1$ or 3, so $p \mid t - 1$ or $t - 3$. If $\varepsilon = 3$ then $3 \mid p - 1$ and $N(P)/C(P) \trianglerighteq SL(2,3)$.*

PROOF. By (2.2), $N(P)_x$ is 2-transitive on the $1 + \varepsilon$ subgroups $P(L)$. Hence, if the lemma does not hold then $\varepsilon = 2$ and $N(P)/C(P)$ induces S_3 on these subgroups. Then (2.6) implies $t = p + 2$. Since $N(P)$ acts irreducibly on P and

$1+t>1+\varepsilon$, $P\neq P(x)$ and hence $P(x)=1$. Thus, G_x acts on the lines through x as a group of degree $p+3$ and order divisible by p^2, which is absurd since $p\neq 3$ here (as $t\neq p^2-p-1$).

COMPLETION OF THE PROOF OF (1.1). By (6.1) and (2.7), $t=2p+3$ and $\varepsilon=3$. Then P has just 2 nontrivial orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of lines on x. Then the commutator group $N(P)'$ fixes $\mathcal{O}_1$ and $\mathcal{O}_2$, and induces a metacyclic group in each $\mathcal{O}_i$, so $N(P)''$ induces the identity on both orbits by (6.1). $N(P)''$ has an element g inverting P. Then g normalizes $P(L)$, so $g\in G(x)$. Now $P=[P,g]\leqq[P,G(x)]\leqq G(x)$, so $1+\varepsilon=1+t$. This contradiction proves the theorem.

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